

# $\mathcal{N} = 8$ Superconformal Chern–Simons Theories

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## Abstract

A Lagrangian description of a maximally supersymmetric conformal field theory in three dimensions was constructed recently by Bagger and Lambert (BL). The BL theory has  $SO(4)$  gauge symmetry and contains scalar and spinor fields that transform as 4-vectors. We verify that this theory has  $OSp(8|4)$  superconformal symmetry and that it is parity conserving despite the fact that it contains a Chern–Simons term. We describe several unsuccessful attempts to construct theories of this type for other gauge groups and representations. This experience leads us to conjecture the uniqueness of the BL theory. Given its large symmetry, we expect this theory to play a significant role in the future development of string theory and M-theory.

# 1 Introduction

Following earlier studies of coincident M2-brane systems [1], Bagger and Lambert (BL) [2, 3] have constructed an explicit action for a new maximally supersymmetric superconformal Chern–Simons theory in three dimensions. The motivation for their work, like that in [4], is to construct the superconformal theories that are dual to  $AdS_4 \times S^7$  solutions of M-theory. Such theories, which are associated to coincident M2-branes, should be maximally supersymmetric, which in three dimensions means that they have  $\mathcal{N} = 8$  supersymmetry. More precisely, the superconformal symmetry group should be  $OSp(8|4)$ , which is also the symmetry of the M-theory solution. It is not obvious that a classical action describing the conformal field theory that is dual to the M-theory solution needs to exist. In fact, there are good reasons to be skeptical: These field theories can be defined as the infrared conformal fixed points of nonconformal  $SU(N)$   $\mathcal{N} = 8$  Yang–Mills theories, but there is no guarantee that any of these fixed points has a dual Lagrangian description.

Ref. [4] attempted to construct three-dimensional theories with  $OSp(8|4)$  superconformal symmetry and  $SU(N)$  gauge symmetry using scalar and spinor matter fields in the adjoint representation of the gauge group. These would be analogous to  $\mathcal{N} = 4$   $SU(N)$  gauge theory in four dimensions, with one crucial difference. The  $F^2$  gauge field kinetic term has the wrong dimension for a conformal theory in three dimensions. Also, it would give propagating degrees of freedom, which are not desired. To address both of these issues, [4] proposed using a Chern–Simons term for the gauge fields instead of an  $F^2$  term. The conclusion reached in [4] was that such an action, with  $\mathcal{N} = 8$  supersymmetry, does not exist. This was consistent with the widely held belief (at the time) that supersymmetric Chern–Simons theories in three dimensions only exist for  $\mathcal{N} \leq 3$ .<sup>1</sup>

The work of Bagger and Lambert [2] presents an explicit action and supersymmetry transformations for an  $\mathcal{N} = 8$  Chern–Simons theory in three dimensions evading the  $\mathcal{N} \leq 3$  bound mentioned above. Their construction can be described in terms of an interesting new type of algebra, which we call a BL algebra.<sup>2</sup> It involves a totally antisymmetric triple bracket analog of the Lie bracket<sup>3</sup>

$$[T^a, T^b, T^c] = f^{abc}_d T^d.$$

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<sup>1</sup>Theories of this type with  $\mathcal{N} = 2$  supersymmetry were first constructed by Ivanov [5] and by Gates and Nishino [6]. For a recent discussion see [7].

<sup>2</sup>Gustavsson, studying the same problem in [8], was independently led to formulate conditions that are equivalent to BL algebras. The equivalence is described in [3].

<sup>3</sup>Such brackets, regarded as generalizations of Poisson brackets, were considered by Nambu in 1973 [9]. For a recent discussion of Nambu brackets see [10].

There should also be a symmetric invertible metric  $h^{ab}$  that can be used to raise and lower indices. The structure constants  $f^{abcd}$  defined in this way are required to have total antisymmetry. Furthermore, this tensor is also required to satisfy a quadratic constraint, analogous to the Jacobi identity, which BL call the “fundamental equation.”

An important question, of course, is whether BL algebras have any nontrivial realizations. BL settle this question by noting that a solution is provided by a set of four generators  $T^a$  that transform as a four-vector of an  $SO(4)$  gauge group. In this example  $f^{abcd} = \varepsilon^{abcd}$  and  $h^{ab} = \delta^{ab}$ . After reviewing the free theory in Section 2, this paper reviews the BL  $SO(4)$  theory in Section 3 making a couple of new observations in the process. The first is an explicit verification that the action is invariant under the conformal supersymmetries as well as the Poincaré supersymmetries. Taken together, these generate the entire  $OSp(8|4)$  symmetry. The second is a careful demonstration in Section 4 of a fact noted in [3], namely that the theory is parity conserving. This feature, which is essential for a dual to the M-theory solution, involves combining a spatial reflection with an  $SO(4) = SU(2) \times SU(2)$  reflection. The latter reflection can be interpreted as interchanging the two  $SU(2)$  factors.

We also explore whether there exist BL theories for other choices of gauge groups and matter representations. Motivated by the  $SO(4)$  example, Section 5 considers parity-conserving theories with gauge group  $G \times G$  and matter fields belonging to a representation  $(R, R)$ , where  $R$  is some representation of  $G$ . Two classes of such examples that have been examined carefully are based on  $G = SO(n)$  and  $G = USp(2n)$  with  $R$  chosen to be the fundamental representation in each case. The first of these two classes is described in detail. The free theory (appropriate for a single M2-brane) appears in this classification as  $G = SO(1)$ , and the  $SO(4)$  theory appears as  $G = USp(2)$ . An invariant totally antisymmetric fourth-rank tensor  $f^{abcd}$ , where  $a, b, c, d$  label components of the representation  $(R, R)$ , can be constructed. However, it turns out that the fundamental equation is satisfied only for the free theory, the  $SO(4)$  theory, and the  $G = SO(2)$  case. The  $SO(2)$  case does not give a new theory, however, for reasons that are explained in the text.

BL suggested that there may be other theories with  $OSp(8|4)$  superconformal symmetry based on nonassociative algebras. Following up on this suggestion, Section 5 attempts to utilize the algebra of octonions in this manner. This leads to a seven-dimensional BL-type algebra. However, once again it turns out that the fundamental identity is not satisfied. Thus, this approach also does not lead to other consistent field theories with  $OSp(8|4)$  superconformal symmetry. Based on these studies, we

conjecture that the  $SO(4)$  BL theory is the only nontrivial three-dimensional Lagrangian theory with  $OSp(8|4)$  superconformal symmetry, at least if one assumes irreducibility and a finite number of fields.

It is a curious coincidence that three-dimensional gravity with a negative cosmological constant can be formulated as a twisted Chern–Simons theory based on the gauge group  $SO(2, 2)$ . Aside from the noncompact form of the gauge group, this is identical to the Chern–Simons term that is picked out by the BL theory. This is discussed in Section 6.

## 2 The Free Theory

Let us start with the well-known free  $\mathcal{N} = 8$  superconformal theory. It contains no gauge fields, so it is not a Chern–Simons theory. The action is

$$S = \frac{1}{2} \int \left( -\partial^\mu \phi^I \partial_\mu \phi^I + i \bar{\psi}^A \gamma^\mu \partial_\mu \psi^A \right) d^3x. \quad (1)$$

This theory has  $OSp(8|4)$  superconformal symmetry. The R-symmetry is  $Spin(8)$  and the conformal symmetry is  $Sp(4) = Spin(3, 2)$ . The index  $I$  labels components of the fundamental  $8_v$  representation of  $Spin(8)$  and the index  $A$  labels components of the spinor  $8_s$  representation. In particular,  $\psi^A$  denotes 8 two-component Majorana spinors. The Poincaré and conformal supersymmetries belong to the other spinor representation,  $8_c$ , whose components are labeled by dotted indices  $\dot{A}$ , etc.

The three inequivalent eight-dimensional representations of  $Spin(8)$  can couple to form a singlet. The invariant tensor (or Clebsch–Gordan coefficients) describing this is denoted  $\Gamma_{AA}^I$ , since it can be interpreted as eight matrices satisfying a Dirac algebra. We also use the transpose matrix, which is written  $\Gamma_{\dot{A}\dot{A}}^I$  without adding an extra symbol indicating that it is the transpose. These matrices have appeared many times before in superstring theory.

Note that in our conventions  $\gamma^\mu$  are  $2 \times 2$  matrices and  $\Gamma^I$  are  $8 \times 8$  matrices. They act on different vector spaces and therefore they trivially commute with one another. BL use a somewhat different formalism in which  $\gamma^\mu$  and  $\Gamma^I$  are 11 anti-commuting  $32 \times 32$  matrices. We find this formalism somewhat confusing, since the three-dimensional theories in question cannot be obtained by dimensional reduction of a higher-dimensional theory (in contrast to  $\mathcal{N} = 4$  super Yang–Mills theory).

The action (1) is invariant under the supersymmetry transformations

$$\delta \phi^I = i \bar{\varepsilon}^{\dot{A}} \Gamma_{\dot{A}A}^I \psi^A = i \bar{\varepsilon} \Gamma^I \psi = i \bar{\psi} \Gamma^I \varepsilon \quad (2)$$

$$\delta\psi = -\gamma \cdot \partial\phi^I \Gamma^I \varepsilon. \quad (3)$$

One can deduce the conserved supercurrent by the Noether method, which involves varying the action while allowing  $\varepsilon$  to have arbitrary  $x$  dependence. This gives

$$\delta S = -i \int \partial_\mu \bar{\varepsilon} \Gamma^I \gamma \cdot \partial\phi^I \gamma^\mu \psi d^3x.$$

Thus the conserved supercurrent is  $i\Gamma^I \gamma \cdot \partial\phi^I \gamma^\mu \psi$ . The conservation of this current is easy to verify using the equations of motion.

Let us now explore the superconformal symmetry. As a first try, let us consider taking  $\varepsilon^{\dot{A}}(x) = \gamma \cdot x \eta^{\dot{A}}$ , since this has the correct dimensions. Using  $\partial_\mu \varepsilon(x) = \gamma_\mu \eta$  and  $\gamma^\mu \gamma^\rho \gamma_\mu = -\gamma^\rho$ , this gives

$$\delta S = i \int \bar{\psi} \gamma \cdot \partial\phi^I \Gamma^I \eta d^3x.$$

This can be canceled by including an additional variation of the form  $\delta\psi \sim \Gamma^I \phi^I \eta$ . Thus the superconformal symmetry is given by

$$\delta\phi^I = i\bar{\psi} \Gamma^I \gamma \cdot x \eta \quad (4)$$

$$\delta\psi = -\gamma \cdot \partial\phi^I \Gamma^I \gamma \cdot x \eta - \phi^I \Gamma^I \eta. \quad (5)$$

One can deduce the various bosonic  $OSp(8|4)$  symmetry transformations by commuting  $\varepsilon$  and  $\eta$  transformations. Of these only the conformal transformation, obtained as the commutator of two  $\eta$  transformations, is not a manifest symmetry of the action. It is often true that scale invariance implies conformal symmetry. However, this is not a general theorem, so it is a good idea to check conformal symmetry explicitly as we have done.

### 3 The $SO(4)$ theory

The  $SO(4)$  gauge theory contains scalar fields  $\phi_a^I$  and Majorana spinor fields  $\psi_a^A$  each of which transform as four-vectors of the gauge group ( $a = 1, 2, 3, 4$ ). In addition there are  $SO(4)$  gauge fields  $A_\mu^{ab}$  with field strengths  $F_{\mu\nu}^{ab}$ . Since four-vector indices are raised and lowered with a Kronecker delta, we do not distinguish superscripts and subscripts.  $A$  and  $F$  are called  $\tilde{A}$  and  $\tilde{F}$  by BL.

The action is a sum of a matter term and a Chern–Simons term:

$$S_k = k (S_m + S_{CS}). \quad (6)$$

We choose normalizations such that the level- $k$  action  $S_k$  is  $k$  times the level-one action  $S_1$ . Then  $k$ , which is a positive integer, is the only arbitrary parameter. Perturbation theory is an expansion in  $1/k$ . So the theory is weakly coupled and can be analyzed in perturbation theory when  $k$  is large. The goal here is to construct and describe the classical action.

The required level-one Chern–Simons action is given by

$$S_{\text{CS}} = \alpha \int \tilde{\omega}_3, \quad (7)$$

where the “twisted” Chern–Simons form  $\tilde{\omega}_3$  is constructed so that

$$d\tilde{\omega}_3 = \frac{1}{2}\epsilon_{abcd}F_{ab} \wedge F_{cd}. \quad (8)$$

This implies that

$$\tilde{\omega}_3 = \frac{1}{2}\epsilon_{abcd}A_{ab} \wedge (dA_{cd} + \frac{2}{3}A_{ce} \wedge A_{ed}). \quad (9)$$

When  $SO(4)$  is viewed as  $SU(2) \times SU(2)$ , this is the difference of the Chern–Simons terms for the two  $SU(2)$  factors. The coefficient  $\alpha$  is chosen so that these have standard level-one normalization. Varying the gauge field by an amount  $\delta A$ , one has (up to a total derivative)

$$\delta\tilde{\omega}_3 = \epsilon_{abcd}\delta A_{ab} \wedge F_{cd}$$

or

$$\delta S_{\text{CS}} = \frac{\alpha}{2} \int \epsilon_{abcd} \epsilon^{\mu\nu\rho} \delta A_{\mu}^{ab} F_{\nu\rho}^{cd} d^3x.$$

The  $SO(4)$  matter action is a sum of kinetic and interaction terms

$$S_{\text{m}} = S_{\text{kin}} + S_{\text{int}}, \quad (10)$$

where

$$S_{\text{kin}} = \int d^3x \left( -\frac{1}{2}(D_{\mu}\phi^I)_a (D^{\mu}\phi^I)_a + \frac{i}{2}\bar{\psi}_a \gamma^{\mu} (D_{\mu}\psi)_a \right) \quad (11)$$

and

$$S_{\text{int}} = \int d^3x \left( ic \epsilon_{abcd} \bar{\psi}_a \Gamma^{IJ} \psi_b \phi_c^I \phi_d^J - \frac{4}{3}c^2 \sum (\epsilon_{abcd} \phi_b^I \phi_c^J \phi_d^K)^2 \right). \quad (12)$$

The supersymmetry transformations that leave the action invariant are

$$\delta\phi_a^I = i\bar{\epsilon}\Gamma^I\psi_a \quad (13)$$

$$\delta\psi_a = -\gamma^{\mu}(D_{\mu}\phi^I)_a \Gamma^I \epsilon + \frac{2c}{3}\epsilon_{abcd}\Gamma^{IJK}\epsilon\phi_b^I \phi_c^J \phi_d^K \quad (14)$$

$$\delta A_{\mu ab} = 4ic \epsilon_{abcd} \bar{\psi}_c \gamma_{\mu} \Gamma^I \phi_d^I \epsilon \quad (15)$$

for the identification

$$c = \frac{1}{16\alpha}.$$

The formulas agree with BL for  $c = 3$ , which corresponds to  $\alpha = 1/48$ . Any apparent minus-sign discrepancies are due to the different treatment of the Dirac matrices discussed earlier.

The conformal supersymmetries also hold. They can be analyzed in the same way that was discussed for the free theory. The result, as before, is to replace  $\varepsilon$  by  $\gamma \cdot x \eta$  and to add a term  $-\phi_a^I \Gamma^I \eta$  to  $\delta\psi_a$ . We have verified the Poincaré and the conformal supersymmetries of this theory in complete detail. Thus this theory has  $OSp(8|4)$  superconformal symmetry and  $SO(4)$  gauge symmetry. It also has parity invariance, which we explain in the next section.

## 4 Parity Conservation

The relative minus sign between the two  $SU(2)$  contributions to the Chern–Simons term has an interesting consequence. Normally, Chern–Simons theories are parity violating. In this case, however, one can define the parity transformation to be a spatial reflection together with interchange of the two  $SU(2)$  gauge groups. Then one concludes that the Chern–Simons term is parity conserving.<sup>4</sup>

To conclude that the entire theory is parity-conserving, there is one other term that needs to be analyzed. It is the one that has the structure

$$\epsilon_{abcd} \bar{\psi}_a \Gamma^{IJ} \psi_b \phi_c^I \phi_d^J.$$

The interchange of the two  $SU(2)$  groups gives one minus sign (due to the epsilon symbol), so invariance will only work if a spinor bilinear of the form  $\bar{\psi}_1 \psi_2 = \psi_1^\dagger \gamma^0 \psi_2$  is a pseudoscalar in three dimensions. So we must decide whether this is true. Certainly, in four dimensions such a structure is usually considered to be a scalar. The R-symmetry labels are irrelevant to this discussion.

Let us review the parity analysis of spinor bilinears in four dimensions. The usual story is that the parity transform (associated to spatial inversion  $\vec{x} \rightarrow -\vec{x}$ ) of a spinor is given by  $\psi \rightarrow \gamma^0 \psi$ . There are two points to be made about this. First, spatial inversion is a reflection in four dimensions. This differs from the case in three-dimensional spacetime, where spatial inversion is a rotation, rather than a reflection.

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<sup>4</sup>This was pointed out to us by A. Kapustin before the appearance of [3]. This way of implementing parity conservation, including the odd parity of a spinor bilinear, was understood already in [11].

Therefore, it is more convenient for generalization to the three-dimensional case to consider a formula for the transformation of a spinor under reflection of only one of the spatial coordinates ( $x^i$ , say). Under this reflection, the formula in four dimensions is  $\psi \rightarrow i\gamma^i\gamma_5\psi$ . For this choice reflecting all three coordinates gives the previous rule  $\psi \rightarrow \gamma^0\psi$  (up to an ambiguous and irrelevant sign). With this rule, one can easily show that  $\bar{\psi}_1\psi_2$  is a scalar and  $\bar{\psi}_1\gamma_5\psi_2$  is a pseudoscalar, as usual.

The second point is that the Dirac algebra for four-dimensional spacetime has an automorphism  $\gamma^\mu \rightarrow i\gamma^\mu\gamma_5$ . In other words,

$$\{i\gamma^\mu\gamma_5, i\gamma^\nu\gamma_5\} = \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

This automorphism squares to  $\gamma^\mu \rightarrow -\gamma^\mu$ , which is also an automorphism. The kinetic term, which involves  $\bar{\psi}\gamma \cdot \partial\psi$ , is invariant under this automorphism, since  $i\gamma^0\gamma_5 i\gamma^\mu\gamma_5 = \gamma^0\gamma^\mu$ . In view of this automorphism, it is equally sensible to define a reflection by the rule  $\psi \rightarrow \gamma^i\psi$ . However, if one makes this choice, then one discovers that  $\bar{\psi}_1\psi_2$  is a pseudoscalar and  $i\bar{\psi}_1\gamma_5\psi_2$  is a scalar. This makes sense, since they (and their negatives) are interchanged by the automorphism.

In the case of three dimensions, there is no analog of  $\gamma_5$ , and so the automorphism discussed above has no analog. As a result, the only sensible rule for a reflection is  $\psi \rightarrow \gamma^i\psi$ . Then one is forced to conclude (independent of any conventions) that  $\bar{\psi}_1\psi_2$  is a pseudoscalar. This is what we saw is required for the  $SO(4)$  super Chern–Simons theory to be parity conserving.

## 5 The Search for Generalizations

Possible generalizations of the  $SO(4)$  theory are suggested by the fact that  $SO(4) = SU(2) \times SU(2) = USp(2) \times USp(2)$  and that a four-vector field  $\phi^a$  can be reexpressed as a bifundamental field  $\phi^{\alpha\alpha'}$ .

An infinite class of candidate theories with the same type of structure is based on the gauge group  $SO(n) \times SO(n)$  with matter fields  $\phi^{\alpha\alpha'}$  assigned to the bifundamental representation  $(\mathbf{n}, \mathbf{n})$ . In this case one takes the gauge field to be

$$A_{\alpha\alpha'\beta\beta'} = \delta_{\alpha\beta}A'_{\alpha'\beta'} + \delta_{\alpha'\beta'}A_{\alpha\beta}, \quad (16)$$

where  $A_{\alpha\beta} = -A_{\beta\alpha}$  and  $A'_{\alpha'\beta'} = -A'_{\beta'\alpha'}$  are  $SO(n)$  gauge fields. The  $n = 1$  case is the free theory with 8 scalars and 8 spinors and no gauge fields, which was discussed in Section 2.



The BL structure constants vanish for  $n = 1$ , and for  $n > 1$  they are given by

$$f^{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'} = \frac{1}{2(n-1)} \left( -\delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\alpha'\delta'}\delta^{\beta'\gamma'} + \delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\alpha'\gamma'}\delta^{\beta'\delta'} \right. \\ \left. -\delta^{\alpha\gamma}\delta^{\delta\beta}\delta^{\alpha'\beta'}\delta^{\gamma'\delta'} + \delta^{\alpha\gamma}\delta^{\delta\beta}\delta^{\alpha'\delta'}\delta^{\gamma'\beta'} - \delta^{\alpha\delta}\delta^{\beta\gamma}\delta^{\alpha'\gamma'}\delta^{\delta'\beta'} + \delta^{\alpha\delta}\delta^{\beta\gamma}\delta^{\alpha'\beta'}\delta^{\delta'\gamma'} \right). \quad (17)$$

For this choice one finds that the dual gauge field is

$$\tilde{A}^{\alpha\alpha'\beta\beta'} = f^{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'} A_{\gamma\gamma'\delta\delta'} = \delta^{\alpha\beta} A'^{\alpha'\beta'} - \delta^{\alpha'\beta'} A^{\alpha\beta}.$$

Therefore the twisted Chern–Simons term again is proportional to the difference of the individual Chern–Simons terms, as required by parity conservation. However, the BL fundamental equation is not satisfied for  $n > 2$ , and there are a number of inconsistencies in the supersymmetry algebra. This leaves the  $n = 2$  case as the only remaining candidate for a new theory. This theory (if it exists) has the same matter content as the BL theory, but fewer gauge fields. Even though the BL algebra is okay in this case, the elimination of four gauge fields gives a violation of another requirement. Specifically, the antisymmetric tensor  $f_{abcd}$  is not  $SO(2) \times SO(2)$  adjoint valued in a pair of indices. This is an essential requirement, because the formula for the supersymmetry variation of the gauge field has the form

$$\delta A_{\mu ab} = 4ic f_{abcd} \bar{\psi}_c \gamma_\mu \Gamma^I \phi_d^I \varepsilon. \quad (18)$$

This equation does not make sense when the right-hand side introduces unwanted degrees of freedom that do not belong to the adjoint representation. This problem arises for all cases with  $n > 1$  including the  $n = 2$  case in particular. One could try to remove the nonadjoint pieces of the right-hand side, but that leads to other inconsistencies.

A completely analogous analysis exists for candidate theories based on the gauge group  $USp(2n) \times USp(2n)$  with matter fields belonging to the bifundamental representation. For the choice  $n = 1$  this is the  $SO(4)$  theory of Section 3. Again, one can construct a totally antisymmetric tensor  $f^{abcd}$  for all  $n$ . However, this does give any new theories, because the BL fundamental equation is not satisfied for  $n > 1$ .

Let us now describe another attempt to construct new examples. BL describe a systematic way to obtain totally antisymmetric triple brackets based on nonassociative algebras. However, the examples they discuss all involve adjoining “a fixed Hermitian matrix  $G$ ” that does not seem to be compatible with a conventional Lie algebra interpretation. Here we explore dispensing with such an auxiliary matrix and applying their procedure to the most familiar nonassociative algebra we know,

namely the algebra of octonions. The question to be addressed is then whether this gives a new superconformal theory with the gauge group  $G_2$  and with the matter fields belonging to the seven-dimensional representation.

Let us denote the imaginary octonions by  $e_a$  with  $a = 1, 2, \dots, 7$ . These have the nonassociative multiplication table

$$e_a e_b = t_{abc} e_c - \delta_{ab}.$$

The totally antisymmetric tensor  $t_{abc}$  has the following nonvanishing components

$$t_{124} = t_{235} = t_{346} = t_{457} = t_{561} = t_{672} = t_{713} = 1.$$

Note that these are related by cyclic permutation of the indices  $(a, b, c) \rightarrow (a+1, b+1, c+1)$ . It is well known that  $t_{abc}$  can be regarded as an invariant tensor describing the totally antisymmetric coupling of three seven-dimensional representations of the Lie group  $G_2$ .

Let  $T_{ab}$  denote a generator of an  $SO(7)$  rotation in the  $ab$  plane. The  $SO(7)$  Lie algebra is

$$[T_{ab}, T_{cd}] = T_{ad}\delta_{bc} - T_{bd}\delta_{ac} - T_{ac}\delta_{bd} + T_{bc}\delta_{ad}.$$

The generators of  $G_2$  can be described as a 14-dimensional subalgebra of this Lie algebra. A possible choice of basis is given by

$$X_1 = T_{24} - T_{56} \quad \text{and} \quad Y_1 = T_{24} - T_{37}$$

and cyclic permutations of the indices. This gives 14 generators  $X_A$  consisting of  $X_a$  and  $X_{a+7} = Y_a$ . By representing the generators  $T_{ab}$  by seven-dimensional matrices in the usual way, one can represent the  $G_2$  generators by antisymmetrical seven-dimensional matrices. These can then be used in the usual way to express  $G_2$  gauge fields as seven-dimensional matrices  $A_{ab}$ .

The group  $G_2$  is a subgroup of  $SO(7)$  in which the **7** of  $SO(7)$  corresponds to the **7** of  $G_2$ . Thus, the seven-index epsilon symbol, which is an invariant tensor of  $SO(7)$ , is also an invariant tensor of  $G_2$ . It can be used to derive an antisymmetric fourth-rank tensor of  $G_2$ :

$$f_{abcd} = \frac{1}{6} \epsilon_{abcdefg} t_{efg}.$$

This tensor has the following nonzero components

$$f_{7356} = f_{1467} = f_{2571} = f_{3612} = f_{4723} = f_{5134} = f_{6245} = 1.$$

These are also related by cyclic permutations. This tensor is the same (up to normalization) as the one given by the construction based on associators that was proposed by BL.

If one defines

$$[abc, def] = \sum_x f_{abcx} f_{defx},$$

the BL fundamental equation takes the form

$$[abw, xyz] - [abx, yzw] + [aby, zwx] - [abz, wxy] = 0.$$

Note that the left-hand side has antisymmetry in the pair  $(a, b)$  and total antisymmetry in the four indices  $(w, x, y, z)$ . One can verify explicitly that these relations are *not* satisfied by the tensor  $f_{abcd}$  given above. (BL did not claim that it necessarily would satisfy the fundamental equation.) Thus, the tensor  $f_{abcd}$  does not define a seven-dimensional BL algebra, and we do not obtain a new theory for the gauge group  $G_2$ .

## 6 Relation to anti de Sitter gravity?

Pure three-dimensional gravity with a negative cosmological constant can be formulated as a twisted Chern–Simons theory based on the gauge group  $SO(2, 2)$ . [12, 13, 14] The BL theory, on the other hand, requires a twisted Chern–Simons term for the gauge group  $SO(4)$ . Aside from the signature, these are exactly the same! What should one make of this coincidence?<sup>5</sup>

The BL theory was motivated by the desire to construct conformal field theories dual to gravity in four-dimensional anti de Sitter space. So the notion that it might be possible to interpret it as a gravity theory in three-dimensional anti de Sitter space is certainly bizarre. The BL theory can be modified easily to the gauge group  $SO(2, 2)$ , though this introduces some disturbing minus signs into half of the kinetic terms of the scalar and spinor fields. If one makes this change anyway, the Chern–Simons term is exactly that for gravity. However, there is a serious problem with a gravitational interpretation in addition to the problem of the negative kinetic terms: a gravity theory should have diffeomorphism symmetry. The Chern–Simons term has this symmetry, but the matter terms in the Lagrangian contain the three-dimensional Lorentz metric to contract indices, so they are not diffeomorphism invariant. Thus, we believe that there is no sensible interpretation of the BL theory as a three-dimensional

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<sup>5</sup>This section was motivated by a question raised by Aaron Bergman at a seminar given by JHS.

gravity theory. Nonetheless, it is striking that its Chern–Simons term is so closely related to the one that arises in the Chern–Simons description of three-dimensional gravity with a negative cosmological constant.

The  $SO(2, 2)$  Chern–Simons formulation of three-dimensional gravity in anti de Sitter space has supergravity generalizations, which can be formulated as Chern–Simons theories for the supergroups [12]

$$OSp(p|2) \times OSp(q|2).$$

The pure gravity case corresponds to  $p = q = 0$ . The existence of these supergravity theories, together with the bizarre coincidence noted above, suggests trying to generalize the BL theory to the corresponding supergroup extensions of  $SO(4)$ . This idea encounters problems with spin and statistics, since the odd generators of this supergroup are not spacetime spinors.

## 7 Conclusion

We have studied classical Lagrangian theories in three dimensions with  $OSp(8|4)$  superconformal symmetry. This symmetry and parity conservation were explicitly verified for the free theory and the Bagger–Lambert  $SO(4)$  theory. A search for further examples of such theories was described. This work led us to conjecture that there are no other such theories, at least if one assumes a finite number of fields.

The relevance of these superconformal Chern–Simons theories to AdS/CFT is an intriguing question. The free theory (associated to a single M2-brane) is presumably dual to the  $AdS_4 \times S^7$  solution with one unit of flux. Based on an analysis of the moduli space of classical vacua, BL proposed in [3] that the  $SO(4)$  theory is dual to  $AdS_4 \times S^7$  with three units of flux, but they do not discuss how to choose the level  $k$ .

To conclude, maximally supersymmetric conformal field theories with a Lagrangian formulation are not common. The BL theory is the first nontrivial example (above two dimensions) since the construction of  $\mathcal{N} = 4$  super Yang–Mills theory over 30 years ago. Thus, we expect that this theory will play a role in the future development of string theory and M-theory, but it is unclear to us what that role will be.

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